ON PROPERTIES OF THE STRESS POTENTIAL OF ELASTIC BODIES

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The reversibility of the strain of ideally elastic bodies results in the requirement for the existence of a specific strain energy, a function of the strain components, which completely characterizes the mechanical properties of the body material. The stresses are connected to this function by the equality

$$\sigma_{ij} = \partial\Phi/\partial\eta_{ij} \tag{0.1}$$

i.e. will be its gradient in six-dimensional strain space.

An increase in the work expended in strain during the change in strain components towards a withdrawal from the natural (unstressed) state of the body is characteristic for real elastic bodies. This property can be called the stability of the material. It makes growth of the strain of solids impossible without an increase in loading on them. Attention was first turned to this property in plasticity theory, where it was formulated by Drucker [1] as one of the fundamental postulates of this theory. Subsequently, the same author [2] noted that this postulate is not specific for plasticity theory, but is a more general statement which should be considered valid for any solid.

For ideally elastic materials the stability requirement is expressed as a requirement for convexity of the stress potential in the six-dimensional strain space η_{ij}. This latter results from the fact that the expression of the Drucker postulate for elastic materials takes the form

$$d\sigma_{ij}d\eta_{ij} = \frac{\partial^2\Phi}{\partial\eta_{kl}\partial\eta_{ij}}\, d\eta_{kl}d\eta_{ij} \geqslant 0 \qquad (i,\, j,\, k,\, l = 1,\, 2,\, 3) \tag{0.2}$$

Equality is possible either in the trivial case of no additional loading or the case of an incompressible elastic material when hydrostatic pressure will be the additional loading. It is hence seen that the quadratic form (0.2) should be sign-definite. The terminology "real elastic body", which is used above, requires some clarification. The behavior of low-molecular solids, particularly metals, is elastic only for small deformations. On the other hand, there exists a class of materials whose behavior corresponds most completely to a theoretical model of an ideally elastic body. These are high-molecular compounds, which behave analogously to incompressible elastic bodies in a highly-elastic state, as numerous tests have shown [3], where the deformation can reach quite large values.

As usual, we take the potential in the form

$$\Phi = \Phi\,(I_1,\, I_2,\, I_3) \tag{0.3}$$

where for polymers it is sufficient to consider Φ a function of the invariants I_2 and I_3 of the strain tensor. The question of the constraints imposed by the condition (0.2) on the potential and its derivatives is considered herein, and the meaning of these constraints is explained.

1. Constraints on the elastic potential and elastic material characteristics resulting from the stability postulate (0.2). Let us introduce the following relationships:

$$I_1 = \varepsilon_{ij}\delta_{ij},\ I_2 = {}^1/_2\,\varepsilon_{ij}\varepsilon_{ij} - {}^1/_6\,I_1^2 \qquad \varepsilon_{ij} = \begin{cases} \eta_{ij} & i=j \\ {}^1/_2\,\eta_{ij} & i\neq j \end{cases} \tag{1.1}$$
$$I_3 = -\,\varepsilon_{ik}\varepsilon_{kj}\varepsilon_{ji} + 2I_1I_2 + {}^1/_9\,I_1^3$$

$$\partial I_1 / \partial\varepsilon_{ij} = \delta_{ij}, \qquad \partial I_2 / \partial\varepsilon_{ij} = \varepsilon_{ij} - {}^1/_3\,I_1\delta_{ij}$$

$$\partial I_3 / \partial\varepsilon_{ij} = -\,3\varepsilon_{ik}\varepsilon_{kj} + 2I_1(\varepsilon_{ij} - {}^1/_3\,I_1\delta_{ij}) + (2I_2 + {}^1/_3\,I_1^2)\delta_{ij} \tag{1.2}$$

Here η_{ij} are the strain components. Taking account of (1.2), we obtain from (0.2)

$$d\sigma_{ij}d\varepsilon_{ij} = \left(\frac{\partial^2\Phi}{\partial I_1^2} - \frac{1}{3}\frac{\partial\Phi}{\partial I_2}\right)(dI_1)^2 + \left(\frac{\partial^2\Phi}{\partial I_1\partial I_2} + 2\frac{\partial\Phi}{\partial I_3}\right)2dI_1dI_2 +$$
$$+ 2\frac{\partial^2\Phi}{\partial I_1\partial I_3}dI_1dI_3 + \left(\frac{\partial^2\Phi}{\partial I_2^2} + 2I_1\frac{\partial^2\Phi}{\partial I_2\partial I_3}\right)(dI_2)^2 +$$
$$+ \frac{\partial^2\Phi}{\partial I_3^2}(dI_3)^2 + \left(\frac{\partial^2\Phi}{\partial I_2\partial I_3} + I_1\frac{\partial^2\Phi}{\partial I_3^2}\right)2dI_2dI_3 +$$
$$+ \left(\frac{\partial\Phi}{\partial I_2} + 2I_1\frac{\partial\Phi}{\partial I_3}\right)d\varepsilon_{kl}d\varepsilon_{kl} - 6\frac{\partial\Phi}{\partial I_3}d\varepsilon_{kl}d\varepsilon_{km}\varepsilon_{lm} \tag{1.3}$$

Let us select a frame of reference such that the coordinate axes would coincide with the principal axes of the tensor ε_{ij}. Then by virtue of (1.2) the differentials dI_k contain only $d\varepsilon_{11}$, $d\varepsilon_{22}$ and $d\varepsilon_{33}$, and the quadratic form will consist only of terms with paired products of these differentials, and with squares of $d\varepsilon_{ij}$. Let A_{11}, A_{22}, ..., A_{66} denote the coefficients of $(d\varepsilon_{11})^2$, $(d\varepsilon_{22})^2$, ..., $(d\varepsilon_{23})^2$, and $2A_{12}$, $2A_{13}$ and $2A_{23}$ of $d\varepsilon_{11}d\varepsilon_{22}$, $d\varepsilon_{11}d\varepsilon_{33}$ and $d\varepsilon_{22}d\varepsilon_{33}$, respectively.

The form (1.3) divides into two: one in the variables $d\varepsilon_{11}$, $d\varepsilon_{22}$, $d\varepsilon_{33}$ and the other in the variables $d\varepsilon_{12}$, $d\varepsilon_{13}$ and $d\varepsilon_{23}$. In conformity with this and the condition of positive definiteness of the forms (1.3)

$$\det\| A_{lm} \| > 0 \qquad (l, m = 1, 2, 3) \tag{1.4}$$

$$D_4 = A_{44}D_3 > 0, \quad A_{44} > 0; \qquad D_5 = A_{55}A_{44}D_3 > 0, \quad A_{55} > 0$$
$$D_6 = A_{66}A_{55}A_{44}D_3 > 0, \quad A_{66} > 0 \tag{1.5}$$

decompose into the positive definiteness condition for each of the forms separately. Namely, (1.4) refers to the former, and (1.5) to the latter. This makes possible a separate analysis of each quadratic form. The additional stress $d\sigma_{ij}$ causes two mutually independent classes of changes $d\varepsilon_{ij}$ of the tensor ε_{ij}: a change in the invariants is the first form, and rotation of the principal axes is the second. Hence, the singularity in the quadratic form (1.3) noted above indeed becomes conceivable.

Let us examine the first form. Let us introduce the following in place of conditions (1.4):

$$(1) \quad A_{11} + A_{22} + A_{33} > 0$$
$$(2) \quad A_{11}A_{22} + A_{11}A_{33} + A_{22}A_{33} - A_{12}^2 - A_{13}^2 - A_{23}^2 > 0 \tag{1.6}$$
$$(3) \quad \det\| A_{kl} \| > 0 \qquad (k, l = 1, 2, 3)$$

It can be shown that these conditions are equivalent to (1.4). To do this it is sufficient to reduce the symmetric matrix of the coefficients of the quadratic form to diagonal form with some elements B_{11}, B_{22}, B_{33}. Then conditions (1.4) are equivalent to the positivity of these elements, and conditions (1.6) become correspondingly

$$B_{11} + B_{22} + B_{33} > 0, \qquad B_{11}B_{22} + B_{11}B_{33} + B_{22}B_{33} > 0, \quad B_{11}B_{22}B_{33} > 0$$

It follows from the first condition that at least one element, B_{11} say, should be positive, and it is clear from this latter that only two (B_{22} and B_{33}) can be negative elements. Let us rewrite the second condition as

$$B = B_{22}(B_{11} + {}^1/_2 B_{33}) + B_{33}(B_{11} + {}^1/_2 B_{22})$$

The quantities in parentheses are positive, hence, the whole expression is negative. Therefore, the assumption on the negativity of B_{22} and B_{33} is incorrect, and (1.4) and (1.6) turn out to be equivalent.

For the frame of reference taken $\varepsilon_{11} = \eta_1$, $\varepsilon_{22} = \eta_2$, $\varepsilon_{33} = \eta_3$, where the η_i are the principal strains connected to the invariants I_k by means of the relationships

$$\eta_1 = {}^2/_3\sqrt{3}I_2^{1/2}\sin(\varphi + {}^2/_3\pi) + {}^1/_3 I_1, \quad \eta_2 = {}^2/_3\sqrt{3}I_2^{1/2}\sin\varphi + {}^1/_3 I_1$$

$$\eta_3 = {}^2/_3\sqrt{3}I_2^{1/2}\sin(\varphi + {}^4/_3\pi) + {}^1/_3 I_1, \quad \varphi = {}^1/_3\arcsin({}^1/_2\sqrt{3}I_3 / I_2^{3/2})$$

$$-{}^1/_6\pi \leqslant \varphi \leqslant {}^1/_6\pi \qquad (1.7)$$

Utilizing (1.1), (1.2) to evaluate the coefficients A_{ij} $(i, j = 1, 2, 3)$, we obtain from (1.6) (1.8)

$$3\partial^2\Phi / \partial I_1^2 + \Psi_1 > 0, \qquad \Psi_1\partial^2\Phi / \partial I_1^2 + \Psi_2 > \Psi_3, \qquad \Psi_2\partial^2\Phi / \partial I_1^2 \geqslant \Psi_4$$

The equality in the last of the relationships (1.8) refers to the case of an incompressible material. In (1.8)

$$\Psi_1 = 2I_2\frac{\partial^2\Phi}{\partial I_2^2} + 6I_3\frac{\partial^3\Phi}{\partial I_2\partial I_3} + 2\frac{\partial\Phi}{\partial I_2} + 6I_2^2\frac{\partial^2\Phi}{\partial I_3^2}$$

$$\Psi_2 = (4I_2^3 - 3I_3^2)\left[\frac{\partial^2\Phi}{\partial I_2^2}\frac{\partial^2\Phi}{\partial I_2^2} - \left(\frac{\partial^2\Phi}{\partial I_2\partial I_3}\right)^2\right] + \frac{2I_2}{3}\frac{\partial^2\Phi}{\partial I_2^2}\frac{\partial\Phi}{\partial I_2} -$$

$$-2I_3\frac{\partial^3\Phi}{\partial I_2^2}\frac{\partial\Phi}{\partial I_2} + 2I_3\frac{\partial^2\Phi}{\partial I_2\partial I_3}\frac{\partial\Phi}{\partial I_2} - 8I_2^2\frac{\partial\Phi}{\partial I_2\partial I_3}\frac{\partial\Phi}{\partial I_3} +$$

$$+2I_2^2\frac{\partial_3\Phi}{\partial I_3^2}\frac{\partial\Phi}{\partial I_2} - 6I_2I_3\frac{\partial^2\Phi}{\partial I_3^2}\frac{\partial\Phi}{\partial I_3} + \frac{1}{3}\left(\frac{\partial\Phi}{\partial I_2}\right)^2 - 4I_2\left(\frac{\partial\Phi}{\partial I_3}\right)^2$$

$$\Psi_3 = 2I_2\left(\frac{\partial^2\Phi}{\partial I_1\partial I_2}\right)^2 + 6I_3\frac{\partial^2\Phi}{\partial I_1\partial I_2}\frac{\partial^2\Phi}{\partial I_1\partial I_3} + 6I_2^2\left(\frac{\partial^2\Phi}{\partial I_1\partial I_3}\right)^2$$

$$\Psi_4 = \left[(4I_2^3 - 3I_3^2)\frac{\partial^2\Phi}{\partial I_3^2} + \frac{2I_2}{3}\frac{\partial\Phi}{\partial I_2} - 2I_3\frac{\partial\Phi}{\partial I_3}\right]\left(\frac{\partial^2\Phi}{\partial I_1\partial I_2}\right)^2 -$$

$$-\left[2(4I_2^3 - 3I_3^2)\frac{\partial^2\Phi}{\partial I_2\partial I_3} - 2I_3\frac{\partial\Phi}{\partial I_2} + 8I_2^2\frac{\partial\Phi}{\partial I_3}\right]\frac{\partial^2\Phi}{\partial I_1\partial I_2}\frac{\partial^2\Phi}{\partial I_1\partial I_3} +$$

$$+\left[(4I_2^3 - 3I_3^2)\frac{\partial^2\Phi}{\partial I_2^2} + 2I_2^2\frac{\partial\Phi}{\partial I_2} - 6I_2I_3\frac{\partial\Phi}{\partial I_3}\right]\left(\frac{\partial\Phi}{\partial I_1\partial I_3}\right)^2 \qquad (1.9)$$

Let us introduce the following generalized elastic moduli according to [4]

$$K = \frac{1}{3}\frac{\sigma}{I_1}, \quad G = \frac{1}{2}\frac{J_2^{1/2}}{I_2^{1/2}}, \quad \omega = \psi - \varphi, \quad -\frac{\pi}{3} \leqslant \omega \leqslant \frac{\pi}{3} \qquad (1.10)$$

They are connected to the potential Φ by means of the relationship

$$\partial\Phi / \partial I_1 = KI_1, \qquad \partial\Phi / \partial I_2 = 2G\cos\omega, \qquad \partial\Phi / \partial\varphi = 4GI_2\sin\omega$$

Here

$$\sigma = \sigma_{ij}\delta_{ij}, \qquad J_2 = {}^1/_2\sigma_{ij}\sigma_{ij} - {}^1/_6\sigma^2, \qquad J_3 = -\sigma_{ik}\sigma_{kj}\sigma_{ji} + 2\sigma J_2 + {}^1/_9\sigma^3$$

$$\psi = {}^1/_3\arcsin({}^1/_2\sqrt{3}J_3 / J_2^{3/2}), \qquad -{}^1/_6\pi \leqslant \psi \leqslant {}^1/_6\pi \qquad (1.11)$$

The potential Φ in (1.9) is considered a function of the invariants I_1, I_2 and I_3, and in (1.11) of the invariants I_1, I_2 and φ. Correspondingly $\partial\Phi / \partial I_2$ has a different meaning in these relations.

It follows from the equality of the derivatives $\partial^2\Phi / \partial I_2 \partial\varphi$ and $\partial^2\Phi / \partial\varphi\partial I_2$ that

$$\cos\omega\, \partial G / \partial\varphi - G \sin\omega\, \partial\omega / \partial\varphi = 2G \sin\omega + \\ + 2I_2 (\sin\omega\, \partial G / \partial I_2 + G \cos\omega\, \partial\omega / \partial I_2) \qquad (1.12)$$

Taking account of the remark relative to $\partial\Phi / \partial I_2$ and utilizing (1.7), (1.11) and (1.12) to evaluate the derivatives in (1.9), we obtain

$$\Psi_1 = -\frac{2}{\sin\omega}\left(2I_2 G \frac{\partial\omega}{\partial I_2} - \frac{\partial G}{\partial\varphi}\right) = \frac{2}{\cos\omega}\left[2I_2 \frac{\partial G}{\partial I_2} + G + G\left(\frac{\partial\omega}{\partial\varphi} + 1\right)\right]$$

$$\Psi_2 = \frac{4G}{3}\left[\left(2I_2 \frac{\partial G}{\partial I_2} + G\right)\left(\frac{\partial\omega}{\partial\varphi} + 1\right) - 2I_2 \frac{\partial G}{\partial\varphi}\frac{\partial\omega}{\partial I_2}\right] \qquad (1.13)$$

$$\Psi_3 = 8I_2\left[\left(\frac{\partial G}{\partial I_1}\right)^2 + \left(G\frac{\partial\omega}{\partial I_1}\right)^2\right]$$

$$\Psi_4 = \frac{16I_2}{3}\left[\left(-\sin\omega \frac{\partial G}{\partial\varphi} + \frac{G}{\cos\omega}\frac{\partial\omega}{\partial\varphi} + 2I_2 \frac{\sin^2\omega}{\cos\omega}\frac{\partial G}{\partial I_2} + \right.\right.$$
$$\left. + G\frac{1+\sin^2\omega}{\cos\omega}\right)\left(\frac{\partial G}{\partial I_1}\right)^2 - 2G\left(\cos\omega \frac{\partial G}{\partial\varphi} - 2I_2 \sin\omega \frac{\partial G}{\partial I_2} - G\sin\omega\right)\frac{\partial G}{\partial I_1}\frac{\partial\omega}{\partial I_1} +$$
$$\left. + G^2\left(\sin\omega \frac{\partial G}{\partial\varphi} + 2I_2 \cos\omega \frac{\partial G}{\partial I_2} + G\cos\omega\right)\left(\frac{\partial\omega}{\partial I_1}\right)^2\right]$$

The function Ψ_4 vanishes either on the deviator axes $\eta_1 = \eta_2 = \eta_3$, when $I_2 = 0$, or for $\partial G / \partial I_1 = \partial\omega / \partial I_1 = 0$. It can be shown that no other cases can vanish.

The expression in parentheses for Ψ_4 is a quadratic form in the quantities $\partial G / \partial I_1$ and $\partial\omega / \partial I_1$. Let L, $2M$ and N denote the coefficients of $(\partial G / \partial I_1)^2$, $(\partial G / \partial I_1)$ $(\partial\omega / \partial I_1)$ and $(\partial\omega / \partial I_1)^2$, respectively. Only for $M^2 - LN \geqslant 0$ can the function Ψ_4 vanish. It is easy to show that $M^2 - LN = -{}^3/_4\Psi_2$. If the discriminant is zero, $\Psi_2 = 0$, and the last inequality in (1.8) is violated. The equality can hold here only for an incompressible material. If $M^2 - LN > 0$, then $\Psi_2 < 0$, hence for compliance with the last inequality in (1.8) it is necessary that $\partial^2\Phi / \partial I_1{}^2 < 0$, and for compliance with the second it is required that $\Psi_1 < 0$, whereupon the first is violated. Other cases of Ψ_4 equalling zero ($L = M = N = 0$; $L = M = 0$, $\partial G / \partial I_1 = 0$, etc.) reduce to those considered above.

Since Ψ_4 is a sign-definite quadratic form, L and N should be of the same sign, and $LN - M^2 > 0$, i.e. $\Psi_2 > 0$. Hence, the sign-definiteness of Ψ_1 follows at once. For example if $\Psi_1 = 0$, we obtain from the first and second relationships in (1.13)

$$2I_2 G \frac{\partial\omega}{\partial I_2} = \frac{\partial G}{\partial\varphi}, \qquad 2I_2 \partial G / \partial I_2 + G = -G(\partial\omega / \partial\varphi + 1)$$

$$\Psi_2 = -{}^4/_3 G^2 [(\partial\omega / \partial\varphi + 1)^2 + 4I_2{}^2 (\partial\omega / \partial I_2)^2] < 0$$

In the two cases noted above of Ψ_4 equalling zero (and Ψ_3, correspondingly), we arrive at the same deductions relative to Ψ_1 and Ψ_2. For example, from the last inequality in (1.8) it follows at once that $\partial^2\Phi / \partial I_1{}^2$ and Ψ_2 should be of the same sign, positive, more exactly, since otherwise either the second inequality (1.8) is violated if

$\Psi_1 > 0$, or the first, if $\Psi_1 < 0$, or both inequalities if $\Psi_1 = 0$. There remains to determine the signs of Ψ_1 and Ψ_4. To do this it is first necessary to explain a number of properties of the function ω and G. We obtain from (1.11)

$$\operatorname{tg}\omega = \frac{1}{2I_2}\left(\frac{\partial\Phi}{\partial\varphi}\right) : \left(\frac{\partial\Phi}{\partial I_2}\right), \quad G = \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial I_2}\right)^2 + \frac{1}{4I_2^2}\left(\frac{\partial\Phi}{\partial\varphi}\right)^2\right]^{1/2}$$

$$\frac{1}{\cos^2\omega}\frac{\partial\omega}{\partial I_2} = \frac{1}{2I_2(\partial\Phi/\partial I_2)^2}\left[\frac{\partial\Phi}{\partial I_2}\left(-\frac{1}{I_2}\frac{\partial\Phi}{\partial\varphi} + \frac{\partial^2\Phi}{\partial I_2\partial\varphi}\right) - \frac{\partial\Phi}{\partial\varphi}\frac{\partial^2\Phi}{\partial I_2^2}\right]$$

$$\frac{1}{\cos^2\omega}\frac{\partial\omega}{\partial\varphi} = \frac{1}{2I_2(\partial\Phi/\partial I_2)^2}\left(\frac{\partial\Phi}{\partial I_2}\frac{\partial^2\Phi}{\partial\varphi^2} - \frac{\partial\Phi}{\partial\varphi}\frac{\partial^2\Phi}{\partial I_2\partial\varphi}\right)$$

$$\frac{\partial G}{\partial\varphi} = \frac{1}{4G}\left(\frac{\partial\Phi}{\partial I_2}\frac{\partial^2\Phi}{\partial I_2\partial\varphi} + \frac{1}{4I_2^2}\frac{\partial\Phi}{\partial\varphi}\frac{\partial^2\Phi}{\partial\varphi^2}\right) \tag{1.14}$$

Using (1.7), we find

$$\frac{\partial\Phi}{\partial\varphi} = 2\sqrt{3}I_2^{3/2}\cos 3\varphi\frac{\partial\Phi}{\partial I_3}$$

$$\frac{\partial^2\Phi}{\partial\varphi^2} = -6\sqrt{3}I_2^{3/2}\sin 3\varphi\frac{\partial\Phi}{\partial I_3} + 12I_2^3\cos^2 3\varphi\frac{\partial^2\Phi}{\partial I_3^2} \tag{1.15}$$

$$\frac{\partial^2\Phi}{\partial I_2\partial\varphi} = 3\sqrt{3}I_2^{1/2}\cos 3\varphi\frac{\partial\Phi}{\partial I_3} + 2\sqrt{3}I_2^{3/2}\cos 3\varphi\left(\frac{\partial^2\Phi}{\partial I_2\partial I_3} + \sqrt{3}I_2^{1/2}\sin 3\varphi\frac{\partial^2\Phi}{\partial I_3^2}\right)$$

In combination with (1.14), these relationships permit making the following deductions relative to ω:

1) For $I_2 = 0$ the quantity $\omega = 0$ since $\partial\Phi / \partial I_2 \neq 0$ because of the continuity of ω, simultaneous disappearance of $^1/_2\, I_2^{-1}\, \partial\Phi / \partial\varphi$ and $\partial\Phi / \partial I_2$ is excluded because $G > 0$;

2) The sign of ω is determined by the sign of $\partial\Phi / \partial\varphi$. This follows from (1.11) and $G > 0$;

3) The quantity ω is a single-valued and bounded function of the invariants, as is seen from (1.10) and the first of relations (1.14).

Analyzing (1.14) and (1.15) jointly, we find that for $I_2 = 0$

$$\partial\omega / \partial\varphi = \partial G / \partial\varphi = I_2\partial\omega / \partial I_2 = 0$$

By virtue of $\Psi_2 > 0$ the positivity of $2I_2\partial G / \partial I_2 + G$ follows hence and from (1.13) the positivity of Ψ_1 and the coefficient N for $I_2 = 0$. correspondingly. By virtue of their sign-constancy the quantities Ψ_1 and Ψ_4 should remain positive in the whole stability domain, and the same deduction relative to $\partial^2\Phi / \partial I_1^2$ follows from the last relation in (1.8). Thus, the relations (1.8) are equivalent to the following:

$$\begin{gathered}\partial^2\Phi / \partial I_1^2 > 0, \quad \Psi_1 > 0, \quad \Psi_2 > 0, \quad \Psi_4 > 0 \\ \Psi_1\partial^2\Phi / \partial I_1^2 + \Psi_2 > \Psi_3, \qquad \Psi_2\partial^2\Phi / \partial I_1^2 > \Psi_4\end{gathered} \tag{1.16}$$

The last two inequalities in (1.16) indicate that the dependences of the functions G and ω on the variable I_1 are determined by how these same functions depend on I_2 and φ and cannot be arbitrary.

The sign of $(\partial G / \partial\varphi)\ (\partial\omega / \partial I_2)$ in the expression for Ψ_2 in (1.13) plays an essential part in the subsequent analysis of the constraints (1.16). If this quantity is nonnegative, then it follows at once from the second and third relation in (1.16) and the first two in (1.13) that

$$\begin{gathered}2I_2\,\partial G / \partial I_2 + G > 0, \qquad \partial\omega / \partial\varphi + 1 > 0 \\ (2I_2\,\partial G / \partial I_2 + G)\ (\partial\omega / \partial\varphi + 1) > 2I_2\,(\partial G / \partial\varphi)\ (\partial\omega / \partial I_2) \geqslant 0\end{gathered} \tag{1.17}$$

Appending the first inequality in (1.16) to these latter and utilizing (1.10), we obtain

$$\frac{\partial\sigma}{\partial I_1} > 0, \quad \frac{\partial J_2^{1/2}}{\partial I_2^{1/2}}\frac{\partial\psi}{\partial\varphi} > 2I_2^{1/2}\frac{\partial J_2^{1/2}}{\partial\varphi}\frac{\partial\psi}{\partial I_2} \geqslant 0, \quad \frac{1}{2}\frac{\partial J_2^{1/2}}{\partial I_2^{1/2}} > 0, \quad \frac{\partial\psi}{\partial\varphi} > 0 \tag{1.18}$$

It is seen from the second inequality in (1.18) that the dependences $J_2^{1/2}$ on ψ and φ on I_2 should be completely definite, which is connected with the fact that J_2 depends on I_2 and ψ on φ. The remaining inequalities in (1.18) require that σ, $J_2^{1/2}$ and ψ be increasing in the appropriate variables.

Let us clarify under which conditions the quantity $(\partial G / \partial\varphi)$ $(\partial\omega / \partial I_2)$ is nonnegative. Let us first note that when the signs of ω and $\partial\omega / \partial I_2$ agree, compliance with the condition $\Psi_1 > 0$ is possible if and only if the signs of $\partial G / \partial\varphi$ and $\partial\omega / \partial I_2$. agree.

For ω as a single-valued, continuous and bounded function of the variables I_2 and φ there will be increasing and decreasing sections in the variable I_2. Let $\omega > 0$ (analogous reasoning will yield the same result for the case $\omega < 0$). The signs of ω and $\partial\omega / \partial I_2$ will agree on the growth section, and then $\partial G / \partial\varphi$ will have the same sign. At the extremum points $\partial\omega / \partial I_2 = 0$. On the decreasing section ω and $\partial\omega / \partial I_2$ have different signs, and the sign of $\partial G / \partial\varphi$ may not agree with the sign of $\partial\omega / \partial I_2$. This means that $\partial\psi / \partial\varphi$ or $\partial J_2^{1/2} / \partial I_2^{1/2}$ can reverse sign.

Because $\psi = \operatorname{arctg} {}^1/_3\sqrt{3}\mu$ and $\nu = \sqrt{3}\ \operatorname{tg}\varphi$, the sign of $\partial\psi / \partial\varphi$ is determined by the sign of $\partial\mu / \partial\nu$. If it is assumed that $\mu = \mu\ (I_2,\ \nu)$ forms a family of smooth curves on the $\mu\nu$ plane, which intersect only at the points $\nu = 0,\ \pm 1$ (as occurs in experiments), then the greatest deviation of $\partial\mu / \partial\nu$ from unity in absolute value will be at these points. A sign reversal of $\partial\mu/\partial\nu$ is possible at these points only when the deviation of the curve from the line $\mu = \nu$ increases, i.e. as $|\omega|$ increases, while the reverse occurs on the decreasing section. Therefore, a change in the sign of $\partial\psi/\partial\varphi$ is not possible for such curves. As regards the tangential modulus $\partial J_2^{1/2} / \partial I_2^{1/2}$, it is known from experimental results that it is always positive. Thus, the inequalities (1.18) will be exact consequences of the first three inequalities in (1.16) either when the signs of ω and $\partial\omega / \partial I_2$ agree, or for those smooth curves $\mu = \mu\ (I_2,\ \nu)$ which intersect only at the points $\nu = 0,\ \pm 1$ in the $\mu\nu$ plane.

For an incompressible material $\partial^2\Phi / \partial I_1^2 = \Psi_3 = \Psi_4 = 0$ only the second and third conditions of (1.16) remain, and our whole discussion remains valid.

Let us consider the constraints connected with the second quadratic form. Using (1.1), (1.2) we find the coefficients A_{44}, A_{55} and A_{66} and write conditions (1.5) as

$$\frac{\partial\Phi}{\partial I_2} + 3\left(\eta_3 - \frac{1}{3}I_1\right)\frac{\partial\Phi}{\partial I_3} > 0, \quad \frac{\partial\Phi}{\partial I_2} + 3\left(\eta_2 - {}^1/_3 I_1\right)\frac{\partial\Phi}{\partial I_3} > 0$$
$$\frac{\partial\Phi}{\partial I_2} + 3(\eta_1 - {}^1/_3 I_1)\frac{\partial\Phi}{\partial I_3} > 0 \tag{1.19}$$

If (1.7) is taken into account, and $\partial\Phi / \partial I_2$ and $\partial\Phi / \partial I_3$ are evaluated while recalling that the derivatives $\partial\Phi / \partial I_2$ have different meanings in (1.11) and (1.19), we obtain from (1.19)

$$\cos\omega - \sin\omega\frac{\sin 3\varphi - 2\sin(\varphi + {}^4/_3\pi)}{\cos 3\varphi} > 0$$

$$\cos\omega - \sin\omega\frac{\sin 3\varphi - 2\sin\varphi}{\cos 3\varphi} > 0$$

$$\cos\omega - \sin\omega\frac{\sin 3\varphi - 2\sin(\varphi + {}^{2}/_{3}\pi)}{\cos 3\varphi} > 0 \qquad (1.20)$$

Ascribing different values to φ it is easy to find the domain of variation of ω by means of these inequalities. It is defined by two lines on the $\omega\varphi$ plane

$$-{}^{1}/_{6}\pi - \varphi < \omega < {}^{1}/_{6}\pi - \varphi \qquad (1.21)$$

The relationships (1.16) and (1.19) represent a complete system of constraints imposed by the stability condition on the elastic potential and its derivatives.

In concluding this section, let us examine application of stability condition in two simple examples. The potential for a Hooke body is

$$\Phi = {}^{1}/_{2}K^{\circ}I_1^2 + 2G^{\circ}I_2, \quad K^{\circ} = E\,/\,3(1 - 2\nu), \quad G = E\,/\,2\,(1 + \nu)$$

(ν is the Poisson ratio).

From the stability condition it follows that $K^{\circ} > 0$, $G^{\circ} > 0$, which is equivalent to the constraint

$$-1 < \nu < {}^{1}/_{2} \qquad (1.22)$$

In our invariants, the potential for a Murnaghan body [6] is

$$\Phi = \text{const} - p_0 I_1 + {}^{1}/_{6}\,(3\lambda + 2\mu)I_1^2 + 2\mu I_2 + {}^{1}/_{27}(9l + n)I_1^3 +$$
$$+ {}^{1}/_{3}\,(6m - n)\,I_1 I_2 - {}^{1}/_{3} n I_3 \qquad (1.23)$$

where p_0 is hydrostatic pressure, λ, μ Lamé coefficients, l, m, n elastic constants.

To be more graphic, we can put $6m = n$ in (1.23). Subjecting such a potential to the conditions (1.16) and (1.19), we obtain

$$3\,(\lambda + 2\mu) + 2I_1\,(9l + n) > 0, \qquad \mu > 0, \qquad 3\mu^2 - n^2 I_2 > 0 \qquad (1.24)$$

$$2\mu - (\eta_1 - {}^{1}/_{3}I_1)n > 0, \qquad 2\mu - (\eta_2 - {}^{1}/_{3}I_1)n > 0, \qquad 2\mu - (\eta_3 - {}^{1}/_{3}I_1)n > 0$$

The limit value I_1 is determined from the first inequality, and I_2 from the last four. We can consider I_1 and I_2 as given, and to seek constraints on l, m, n. For example, combining the first and second inequalities for small I_1, we obtain constraints on ν which agree with (1.22). If (1.7) is taken into account in (1.24), it is seen that compliance with the third assures compliance with the last three inequalities in (1.24). We hence obtain

$$-\frac{\mu\sqrt{3}}{I_2^{1/2}} < n < \frac{\mu\sqrt{3}}{I_2^{1/2}} \qquad (1.25)$$

Knowing the limits of variation of n, μ, λ the limits in the variation of the last constant l is easily found from the first relationship in (1.24).

2. On the role of the nonlinear tensor terms in the relationship (0.1). As is known, the specific gravity of the nonlinear tensor terms in the stress-strain relations is determined by the quantity ω. We seek the range of variation of ω from the last stability condition in (1.18). According to [5], the following relationship between the Lode parameters μ and ν is valid:

$$\mu = \nu \frac{6(3+\nu^2)^2 - \theta(9-\nu^2)(3-\nu)}{6(3+\nu^2)^2 - 2\theta\nu^2(9-\nu)} \tag{2.1}$$

Here θ is some function of I_2 and ν which characterizes the deviation of the curve $\mu = \mu\,(I_2, \nu)$ from the straight line $\mu = \nu$. It is seen from (2.1) that the curves $\mu = \mu\,(\theta, \nu)$ in the $\mu\nu$ plane absolutely pass through the origin of the points (1, 1) and (–1, –1), which correspond to the origin and the points $({}^1/_6\pi, {}^1/_6\pi)$ and $(-{}^1/_6\,\pi, -{}^1/_6\pi)$ on the $\psi\varphi$ plane. From here and from the last inequality in (1.18) it follows at once that the curves $\psi = \psi\,(I_2, \varphi)$ should be found in the first and third quadrants and $|\ \omega\ | < {}^1/_6\pi$.

The range of variation θ equals $(-3, {}^{16}/_3)$ in [5]. Let us note that for some values of θ from this range μ and ν (ψ and φ, respectively) would have different signs, which is not possible for a stable material.

For example, if $\nu = 0.1$, it is sufficient to take $\theta > 2.05$, and μ will be negative.

The fourth condition in (1.18) is equivalent to positivity of the derivative $\partial\mu\ /\ \partial\nu$. Determination of the range of variation of θ from (2.1) by using the last condition turns out to be possible only under the assumption of independence of θ from ν. Hence, (2.1) generates a one-parameter family of smooth curves on the $\mu\nu$ plane, which intersects only at the origin and at the points (1, 1), (–1, –1). The meaning of the assumption considered earlier in analyzing the stability condition now becomes understandable.

We find $\partial\mu\ /\ \partial\nu$ from (2.1). We equate the numerator of the expression obtained to zero

$$2\nu^2(-\nu^6 + 15\nu^4 - 27\nu^2 - 243)\theta^2 + 6(\nu^2+3)(\nu^6 - 99\nu^4 + 243\nu^2 - \\ -81)\theta + 36(\nu^2+3)^4 = 0 \tag{2.2}$$

Assigning a number of values to ν, we find the roots θ_1 and θ_2 in the form of two curves; the domain between them will be the stability domain for different ν.

Let us present several such points

$$\nu \to 0, \quad \nu = \pm\,0.32, \quad \nu = \pm\,0.63, \quad \nu = \pm\,0.89, \quad \nu = \pm\,1.00$$

$$\theta_1 \to -\infty, \quad \theta_1 = -\,24.50, \quad \theta_1 = -\,4.94, \quad \theta_1 = -\,3.01, \quad \theta_1 = -\,3.00$$

$$\theta_2 \to 2, \quad \theta_2 = 2.70, \quad \theta_2 = 4.94, \quad \theta_2 = 5.99, \quad \theta_2 = 6.00$$

From these data the required interval is defined as

$$-3 < \theta < 2 \tag{2.3}$$

Presented in Fig. 1 are curves of θ_1 and θ_2, and the curves $\mu = \mu\,(\nu)$ for $\theta = 2$, $\theta = -\,3$ in Fig. 2. It is seen that the ultimate state for the first curve sets in for $\nu = +\,0$, and for $\nu = 1 - 0$ for the second. The followinf range in ω

$$|\ \omega\ | < 13^\circ \tag{2.4}$$

corresponds to the range (2.3).

We reconstruct the curves $\mu = \mu\,(\nu)$ for $\theta = 2$ and $\theta = -\,3$ into $\omega = \omega\,(\varphi)$ curves in the $\omega\varphi$ plane. We superpose the straight lines $\omega = {}^1/_6\pi - \varphi$ and $\omega = -\,{}^1/_6\pi - \varphi$ on the curves obtained (Fig. 3). It is seen that the straight lines lie above and below the curves, hence the range of variation of ω is defined entirely by these two curves. This means that compliance with (1.16) implies compliance with

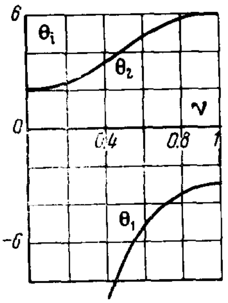

Fig. 1

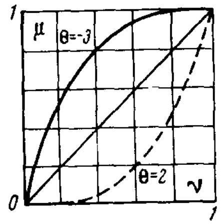

Fig. 2

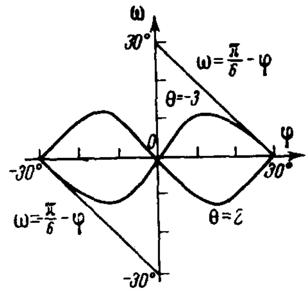

Fig. 3

(1.19) for a simultaneous change in the invariants and a rotation of the principal axes of the tensor ε_{ij}, while the reverse is incorrect. Hence, conservation of (1.16) is necessary and sufficient for a material to be stable.

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